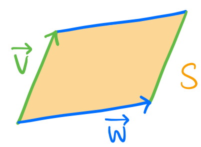
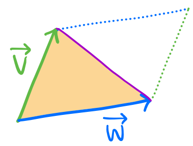
Lecture 12. Determinants as areas and volumes

Prop If A is a 2×2 matrix, $|\det(A)|$ is equal to the area of the parallelogram given by its columns.

$$A = \begin{array}{c} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ \begin{matrix} \vec{v} & \vec{w} \end{matrix} \end{array} \Longrightarrow |\det(A)| = |ad-bc| = \mathrm{Area}(S)$$

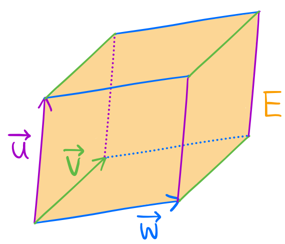


Note For a triangular region with two sides given by the columns, the area is $\frac{1}{2}|\det(A)| = \frac{1}{2}|ad-bc|$



Prop If A is a 3×3 matrix, $|\det(A)|$ is equal to the volume of the parallelopiped given by its columns.

$$A = \begin{array}{c} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ \begin{matrix} \vec{u} & \vec{v} & \vec{w} \end{matrix} \end{array} \Longrightarrow |\det(A)| = \mathrm{Vol}(E)$$

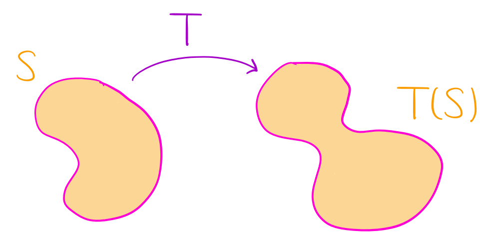


Thm Let $T: \mathbb{R}^n \longrightarrow \mathbb{R}^n$ be a linear transformation with standard matrix A.

(1) For a region S in $\mathbb{R}^n$ with area,

$$\mathrm{Area}(T(S)) = |\det(A)|\,\mathrm{Area}(S)$$

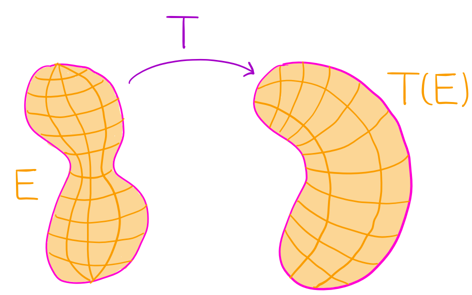
where $T(S)$ denotes the image of S.



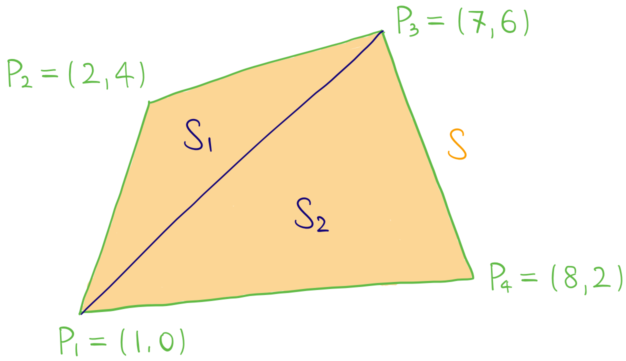
(2) For a region S in $\mathbb{R}^n$ with volume,

$$\mathrm{Vol}(T(E)) = |\det(A)|\,\mathrm{Vol}(E)$$

where $T(E)$ denotes the image of E.



Ex Consider the following region S in $\mathbb{R}^2$.



(1) Find the area of S.

Sol Divide S into two triangular regions S_1 and S_2.

S_1 has two sides given by

$$\overrightarrow{P_1P_2} = \begin{bmatrix} 2-1 \\ 4-0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \quad \overrightarrow{P_1P_3} = \begin{bmatrix} 7-1 \\ 6-0 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}.$$

$$\Rightarrow \text{Area}(S_1) = \frac{1}{2}\left|\det\begin{bmatrix} 1 & 6 \\ 4 & 6 \end{bmatrix}\right| = \frac{1}{2}|1\cdot 6 - 4\cdot 6| = 9$$

S_2 has two sides given by

$$\overrightarrow{P_1P_3} = \begin{bmatrix} 7-1 \\ 6-0 \end{bmatrix} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}, \quad \overrightarrow{P_1P_4} = \begin{bmatrix} 8-1 \\ 2-0 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}$$

$$\Rightarrow \text{Area}(S_2) = \frac{1}{2}\left|\det\begin{bmatrix} 6 & 7 \\ 6 & 2 \end{bmatrix}\right| = \frac{1}{2}|6\cdot 2 - 7\cdot 6| = 15$$

$$\text{Area}(S) = \text{Area}(S_1) + \text{Area}(S_2) = 9 + 15 = \boxed{24}$$

(2) For a linear transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with standard matrix

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix},$$

find the area of $T(S)$.

Sol $\text{Area}(T(S)) = |\det(A)|\,\text{Area}(S) = |3\cdot 6 - 4\cdot 5|\cdot 24 = \boxed{48}$

Ex Find the area of the ellipse given by $\frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1$ with $a, b \in \mathbb{R}$.

Sol We have $\frac{x^2}{a^2}+\frac{y^2}{b^2} \le 1 \Rightarrow \left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2 \le 1$

Set $x' = \frac{x}{a}$ and $y' = \frac{y}{b} \Rightarrow (x')^2+(y')^2 \le 1$: disk of radius 1

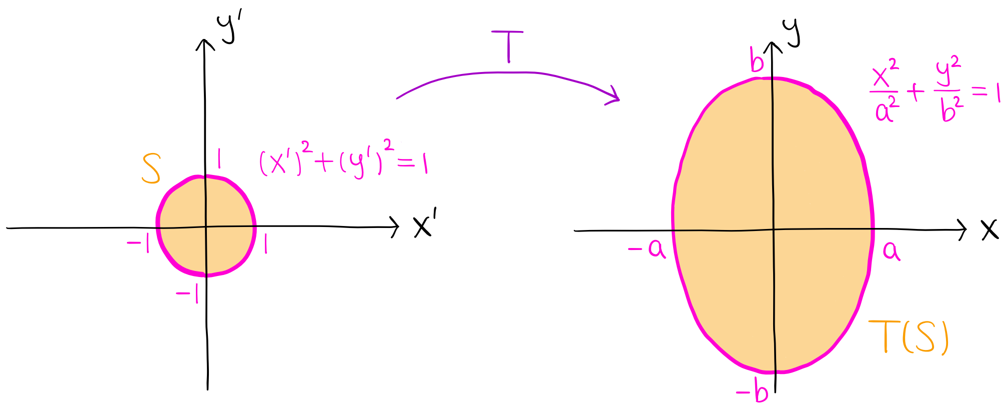
$x = ax', \ y = by' \Rightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} x' \\ y' \end{bmatrix}$

Let S be the disk of radius 1 centered at the origin.

$\Rightarrow$ The ellipse is the image of S under the linear transformation $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ with standard matrix

$$A = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

* T stretches each vector horizontally by a factor a and vertically by a factor b



$\Rightarrow \text{Area}(T(S)) = |\det(A)|\, \text{Area}(S) = |a \cdot b - 0 \cdot 0| \cdot \pi \cdot 1^2 = \boxed{ab\pi}$

Note We can similarly show that the volume of the ellipsoid given by $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} \le 1$ with $a, b, c \in \mathbb{R}$ is $\frac{4}{3}abc\pi$.